J. Fluid Mech. (1978), *vol.* 86, *part* 4, *pp.* 673–693
Printed in Great Britain

The linear stability of the flow in a narrow spherical annulus

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(Received 23 November 1976 and in revised form 29 September 1977)

The flow of a fluid in a narrow spherical annulus is considered. When the outer sphere remains fixed and the angular velocity of the inner one is increased beyond a critical value an instability resembling Taylor vortices appears. This instability is investigated by expanding the solution in powers of the small parameter ϵ, the ratio of the gap thickness to the radius, and assuming that two length scales, $O(1)$ and $O(\epsilon)$, are important in the latitudinal direction.

The perturbation then takes the form of cells which are of roughly square cross-section, at least near the equator, but whose amplitude decays rapidly with latitude; it is also subject to a slow spatial modulation. The critical value of the Taylor number at which the instability first appears is shown to be that for infinite concentric cylinders plus a correction $O(\epsilon)$ due to secondary motions and a correction not greater than $O(\epsilon)$ due to the domain being bounded.

1. Introduction

The linear stability of the flow between infinitely long, differentially rotating concentric circular cylinders is now well known and well understood. If the speed of the inner cylinder is increased above a critical value the flow becomes unstable and takes the form of regular cellular toroidal vortices, called Taylor vortices. In recent years some attention has been devoted to the effect of end walls on this and the similar Bénard convection flow. The presence of end walls makes separation of the variables difficult and poses great mathematical problems. Pellew & Southwell (1940) were the first to circumvent these difficulties by allowing a slip condition on the end walls and Davis (1967) has solved the Bénard problem numerically. More recently Drazin (1975), Hall & Walton (1977), Segel (1969) and Daniels (1977) have made progress with the Bénard problem with free surfaces, in which the variables are separable.

In this paper we consider a geometry which, although the variables are still not separable, does allow some progress to be made analytically. The fluid fills the (narrow) gap between two concentric spheres and is set in motion by the rotation of the inner one. When its speed is sufficiently high, instability is observed to set in as cellular vortices, similar to Taylor vortices, near the equator (Sawatzki & Zierep 1970; Wimmer 1976) and as the speed is further increased vortices appear at higher latitudes. Furthermore, the critical value of the dimensionless parameter governing the flow, the Taylor number T, is observed to be very close to that for infinite concentric cylinders, at least when the gap between the spheres is narrow.

The linear stability of this flow has been investigated numerically by Bratukhin (1961) and Munson & Menguturk (1975) for arbitrary gap widths. Both these studies rely heavily on expansions in spherical harmonics. Munson & Joseph (1971*b*) have discussed the stability by an energy method.

In none of these studies has the close similarity with Taylor vortices been exploited and we seek to do that in this paper. We suppose that $\epsilon \equiv (R_2 - R_1)/R_1 \ll 1$, where R_1 and R_2 are the radii of the inner and outer spheres, and seek the value of the Taylor number $T \equiv R_1 \Omega_1^2 (R_2 - R_1)^3/\nu^2$ at which the flow first becomes unstable. Here Ω_1 is the angular velocity of the inner sphere and ν the kinematic viscosity of the fluid. To a first approximation in ϵ we show that the critical value of T at which the cellular motion first occurs is the same as that for infinite concentric cylinders.

The unperturbed flow is discussed in § 2. As in Munson & Joseph (1971*a*), we assume that a Reynolds number R_M is small and expand in powers of R_M. Here we define $R_M \equiv \epsilon^2 Re$, where $Re = \Omega_1 R_1^2/\nu$ is Munson & Joseph's Reynolds number. Then $\epsilon T = R_M^2$ and the restriction that T be $O(1)$ requires that R_M is $O(\epsilon^{\frac{1}{2}})$. Furthermore, by assuming that $\epsilon \ll 1$ we may expand in powers of ϵ also and avoid expanding in spherical harmonics. To a first approximation in R_M and ϵ the basic flow is meridional and its profile is a simple radial shear whose magnitude decreases with latitude. It may be expected then, as several authors have already suggested, that the neighbourhood of the equator is the most unstable region and that, since $\epsilon \ll 1$, the curvature of the container plays only a minor role and the flow is similar to that between infinite concentric cylinders.

If the instability sets in as toroidal cells of roughly square cross-section, then two length scales in the latitudinal direction, $O(1)$ and $O(\epsilon)$, are likely to be important. Such a structure is amenable to description by a WKB or multiple-scale analysis. In § 3 we formulate the linear stability problem in this way by writing the perturbed angular velocity $\bar{\Omega}$ as

$$\bar{\Omega} = A_0(\theta)\, g_0(\theta, x) \exp\left\{\frac{i}{\epsilon}\int_{\frac{1}{2}\pi}^{\theta} k(\theta)\, d\theta\right\} e^{\sigma t} + O(\epsilon) + \text{complex conjugate} \qquad (1.1)$$

and the Taylor number T as

$$T = T_0 + \epsilon T_1 + \epsilon^2 T_2 + \dots .$$

Here x is a radial variable, θ is the co-latitude, t is the time and σ the growth rate. An eigenvalue problem for the wavenumber k is obtained in § 4 which must be solved numerically for each value of θ and a prescribed value of T_0; there are six solutions.

Close to the equator the solution (1.1) breaks down because of the coincidence of two pairs of eigenvalues and a new scaling is needed. The solution in the region $\frac{1}{2}\pi - \theta \sim \epsilon^{\frac{2}{3}}$ is given in § 5 in terms of Airy functions. This solution in turn breaks down and an inner region of thickness ϵ is needed where the $O(\epsilon)$ correction to the basic flow is even less dominant.

A new scaling near the poles $\theta = 0, \pi$ and the boundary conditions at $\theta = 0, \pi$ are treated in § 6. A summary and a discussion of the results are given in § 7.

2. The undisturbed flow

In this section we consider the steady flow set up in the gap between two concentric spheres of radii R_1 and R_2 ($R_1 < R_2$) when the gap is filled with a viscous incompressible fluid and the inner sphere is rotated with angular velocity Ω_1.

In terms of spherical polar co-ordinates (r^*, θ, ϕ), the assumption that the flow is axisymmetric is equivalent to ignoring longitudinal derivatives. The dimensional velocity components ($v_r^*, v_\theta^*, v_\phi^*$) may then be written in terms of a dimensionless stream function ψ and circulation Ω as follows:

$$v_r^* = (R_2 - R_1)\frac{\Omega_1\, \partial\psi/\partial\theta}{r^2 \sin\theta}, \quad v_\theta^* = -(R_2 - R_1)\frac{\Omega_1\, \partial\psi/\partial r}{r\sin\theta}, \quad v_\phi^* = R_1\Omega_1\frac{\Omega}{r\sin\theta}.$$

Here $r = r^*/R_1$ is a dimensionless radius. As we shall be concerned only with the flow in a narrow annulus, it is convenient to introduce a radial variable x scaled on the gap width. Thus we write

$$r = 1 + \epsilon x,$$

where $\epsilon = (R_2 - R_1)/R_1$ is a dimensionless measure of the gap width.

In terms of ψ and Ω the Navier–Stokes equations are (Rosenhead 1963, p. 132)

$$\frac{-R_M(\psi_x\Omega_\theta - \psi_\theta\Omega_x)}{(1+\epsilon x)^2 \sin\theta} = \tilde{D}^2\Omega, \tag{2.1}$$

$$\frac{2\Omega(\Omega_x(1+\epsilon x)\cos\theta - \epsilon\sin\theta\,\Omega_\theta)}{(1+\epsilon x)^3\sin^2\theta} - \frac{\psi_x(\tilde{D}^2\psi)_\theta - \psi_\theta(\tilde{D}^2\psi)_x}{(1+\epsilon x)^2\sin\theta} + \frac{2\tilde{D}^2\psi((1+\epsilon x)\cos\theta\,\psi_x - \epsilon\sin\theta\,\psi_\theta)}{(1+\epsilon x)^3\sin^2\theta} = R_M^{-1}\tilde{D}^4\psi, \tag{2.2}$$

where

$$\tilde{D}^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\epsilon^2}{(1+\epsilon x)^2}\left[\frac{\partial^2}{\partial\theta^2} - \cot\theta\frac{\partial}{\partial\theta}\right]. \tag{2.3}$$

Here suffixes x and θ denote partial derivatives with respect to x and θ. In the modified Reynolds number $R_M = \Omega_1(R_2 - R_1)^2/\nu$, ν is the kinematic viscosity and $R_2 - R_1$ and $\Omega_1(R_2 - R_1)$ are characteristic lengths and velocities. R_M is related to the more usual Reynolds number $Re = \Omega_1 R_1^2/\nu$ by $R_M = \epsilon^2 Re$.

The specification of the problem is completed with the no-slip conditions at the inner and outer spherical boundaries, i.e.

$$\left.\begin{aligned} \psi_\theta = \psi_x = 0 \quad \text{at} \quad x = 0, 1; \\ \Omega = \sin^2\theta \quad \text{at} \quad x = 0; \quad \Omega = 0 \quad \text{at} \quad x = 1. \end{aligned}\right\} \tag{2.4}$$

As in Munson & Joseph's (1971*a*) treatment of the flow, we shall assume that the appropriate Reynolds number is small, but we shall make the further assumption that the gap width ϵ is small also. ψ and Ω may then be expanded in powers of R_M and ϵ as follows:

$$\left.\begin{aligned} \psi &= R_M(\psi_{00} + R_M^2\psi_{10} + \epsilon\psi_{01} + O(R_M^4, \epsilon^2, R_M^2\epsilon)), \\ \Omega &= \Omega_{00} + R_M^2\Omega_{10} + \epsilon\Omega_{01} + O(R_M^4, \epsilon^2, R_M^2\epsilon). \end{aligned}\right\} \tag{2.5}$$

Munson & Joseph (1971*a*) indicate that the leading terms (as far as R_M^2) give an accurate representation of the solution for $R_M \lesssim 10$. If the gap width is small, say $\epsilon \approx 10^{-3}$, this is equivalent to $Re \lesssim 10^5$, and the Taylor number $T \lesssim 10^5$ (see § 3), which is sufficiently large for our purposes.

The major simplification afforded by taking $\epsilon \ll 1$ is now clear: derivatives in θ contain powers of R_M or ϵ, so that on substituting the expansion (2.3) into (2.2) and equating coefficients of powers of R_M and ϵ we effectively obtain only ordinary differential equations.

Thus we have

$$\partial^2\Omega_{00}/\partial x^2 = 0$$

with $\Omega_{00} = \sin^2\theta$ at $x = 0$ and $\Omega_{00} = 0$ at $x = 1$. Hence we have

$$\Omega_{00} = (1-x)\sin^2\theta. \tag{2.6}$$

Also

$$\frac{\partial^4\psi_{00}}{\partial x^4} = 2\Omega_{00}\frac{\partial\Omega_{00}}{\partial x}\frac{\cos\theta}{\sin^2\theta}$$

with $\psi_{00} = \psi_{00x} = 0$ at $x = 0, 1$. Hence

$$\psi_{00} = -2a_{00}(x)\sin^2\theta\cos\theta, \tag{2.7a}$$

where

$$a_{00}(x) = x^2(1-x)^2(3-x)/5!. \tag{2.7b}$$

Equating terms in ϵ we obtain

$$\frac{\partial^2\Omega_{01}}{\partial x^2} = 0, \quad \frac{\partial^4\psi_{01}}{\partial x^4} = -\frac{4\Omega_{00}}{\sin^2\theta}\left(x\cos\theta\frac{\partial\Omega_{00}}{\partial x} + \tfrac{1}{2}\sin\theta\frac{\partial\Omega_{00}}{\partial\theta}\right)$$

with $\Omega_{01} = \psi_{01} = \psi_{01x} = 0$ at $x = 0, 1$. Hence

$$\Omega_{01} \equiv 0, \tag{2.8}$$

$$\psi_{01} = -a_{01}(x)\sin^2\theta\cos\theta, \tag{2.9a}$$

where

$$a_{01}(x) = 8x^2(x-1)^3(2x-3)/6!. \tag{2.9b}$$

Equating coefficients of R_M^2 we obtain

$$\frac{\partial^2\Omega_{10}}{\partial x^2} = -\frac{(\psi_{00x}\Omega_{00\theta} - \psi_{00\theta}\Omega_{00x})}{\sin\theta},$$

$$\frac{\partial^4\psi_{10}}{\partial x^4} = \frac{2\cos\theta}{\sin^2\theta}(\Omega_{00}\Omega_{10x} + \Omega_{10}\Omega_{00x}) - \frac{(\psi_{00x}\psi_{00xx\theta} - \psi_{00\theta}\psi_{00xxx})}{\sin\theta} + \frac{2\cos\theta}{\sin^2\theta}(\psi_{00xx}\psi_{00x})$$

with $\Omega_{10} = \psi_{10} = \psi_{10x} = 0$ at $x = 0, 1$. After some algebra we obtain

$$\Omega_{10} = b_{101}(x)\sin^2\theta + b_{102}(x)\sin^4\theta, \tag{2.10a}$$

where

$$\left.\begin{aligned} b_{101}(x) &= 4x(x-1)(x-3)(2x-3)(10x^3 - 15x^2 + 6x + 3)/5\times 7!,\\ b_{102}(x) &= -x(x-1)(10x^5 - 60x^4 + 123x^3 - 102x^2 + 18x + 18)/5\times 6!,\end{aligned}\right\} \tag{2.10b}$$

and

$$\psi_{10} = a_{101}(x)\cos\theta\sin^2\theta + a_{102}(x)\cos\theta\sin^4\theta, \tag{2.11a}$$

where

$$\left.\begin{aligned} a_{101}(x) &= 96x^2(x-1)^2(530x^7 - 4770x^6 + 15285x^5 - 21585x^4\\ &\qquad + 10647x^3 + 5457x^2 - 6861x - 1359)/5!11!,\\ a_{102}(x) &= -96x^2(x-1)^2(70x^7 - 630x^6 + 2520x^5 - 5220x^4\\ &\qquad + 4860x^3 + 1080x^2 - 4779x - 243)/5!\,11!.\end{aligned}\right\} \tag{2.11b}$$

We note that, at least to this order, Ω is symmetric and ψ antisymmetric about the equator. In other words the radial and meridional velocities are symmetric and the azimuthal velocity antisymmetric; consequently there is no flow across the equator.

3. The formulation of the linear stability problem

We now consider the linearized stability of the basic flow presented in § 2. We suppose that there is a small perturbation of the basic flow in the form

$$\psi_{\text{total}} = \psi + \Delta\bar{\psi}, \quad \Omega_{\text{total}} = \Omega + \Delta\bar{\Omega},$$

where (ψ, Ω) is the basic solution, $(\bar{\psi}, \bar{\Omega})$ is a perturbation and Δ a small parameter. This is substituted into (2.1) and terms quadratic in Δ ignored. Of course, $\bar{\psi}$ and $\bar{\Omega}$ are functions of two spatial variables x and θ, so that this results in a pair of partial differential equations, viz.

$$(\tilde{D}^2 - \sigma)\,\bar{\Omega} = -(\epsilon T)^{\frac{1}{2}}\left[\frac{\psi_x \bar{\Omega}_\theta - \psi_\theta \bar{\Omega}_x + \bar{\psi}_x \Omega_\theta - \bar{\psi}_\theta \Omega_x}{(1+\epsilon x)^2 \sin\theta}\right], \tag{3.1}$$

$$\begin{aligned}\tilde{D}^2(\tilde{D}^2 - \sigma)\,\bar{\psi} = {} & \frac{2(\epsilon T)^{\frac{1}{2}}}{\sin^2\theta(1+\epsilon x)^3}\left[\cos\theta(1+\epsilon x)\,(\Omega\bar{\Omega}_x + \bar{\Omega}\Omega_x) - \epsilon\sin\theta(\Omega\bar{\Omega}_\theta + \bar{\Omega}\Omega_\theta)\right] \\ & - \frac{(\epsilon T)^{\frac{1}{2}}}{\sin\theta(1+\epsilon x)^2}\left[\psi_x(\tilde{D}^2\bar{\psi})_\theta + \bar{\psi}_x(\tilde{D}^2\psi)_\theta - \psi_\theta(\tilde{D}^2\bar{\psi})_x - \bar{\psi}_\theta(\tilde{D}^2\psi)_x\right] \\ & + \frac{2(\epsilon T)^{\frac{1}{2}}}{\sin^2\theta(1+\epsilon x)^3}\{\cos\theta(1+\epsilon x)\,[(\tilde{D}^2\psi)\,\bar{\psi}_x + (\tilde{D}^2\bar{\psi})\,\psi_x] \\ & - \epsilon\sin\theta[(\tilde{D}^2\psi)\,\bar{\psi}_\theta + (\tilde{D}^2\bar{\psi})\,\psi_\theta]\}. \end{aligned} \tag{3.2}$$

We have assumed a time dependence of the form $\exp\{\sigma t\}$, where σ is a dimensionless growth rate, and we have written $R_M = (\epsilon T)^{\frac{1}{2}}$, where $T = R_1\Omega_1^2(R_2 - R_1)^3/\nu^2$ is the Taylor number.

At the rigid boundaries we must satisfy the no-slip conditions, i.e.

$$\bar{\psi} = \bar{\psi}_x = \bar{\Omega} = 0 \quad \text{at} \quad x = 0, 1. \tag{3.3}$$

In addition we require that

$$v_\theta = v_\phi = 0, \quad v_r \quad \text{finite at} \quad \theta = 0, \pi,$$

and this means that

$$\bar{\psi} \sim \theta^2 \quad \text{for} \quad \theta \ll 1; \quad \bar{\psi} \sim (\pi - \theta)^2 \quad \text{for} \quad \pi - \theta \ll 1. \tag{3.4}$$

Equations (3.1) and (3.2) may be simplified by making some assumptions about the form of $\bar{\psi}$ and $\bar{\Omega}$. Variations in the radial direction have already been assumed to take place on a scale $O(\epsilon)$. If the instability occurs as Taylor vortices whose amplitude varies with θ as experiments (Sawatzki & Zierep 1970; Wimmer 1976) lead us to believe, then we may expect that scales $O(\epsilon)$ and $O(1)$ in θ will be important. A multiple-scaling or WKB formulation is then called for and we may look for solutions in the form

$$(\bar{\psi}, \bar{\Omega}) = A_0(\theta)\, e^{\sigma t} \exp\left\{\frac{i}{\epsilon}\int_{\frac{1}{2}\pi}^{\theta} k(\theta)\, d\theta\right\} ((\epsilon T)^{\frac{1}{2}} f_0(\theta, x), g_0(\theta, x)) + O(\epsilon). \tag{3.5}$$

We shall see later that six possibilities for $k(\theta)$ and $A_0(\theta)$ emerge in principle, so that the solution which satisfies the six conditions at the poles and equator requires summation over all six possibilities. The basic flow must also be expanded in powers of ϵ; from (2.5) we have

$$\left.\begin{aligned}\psi &= (\epsilon T)^{\frac{1}{2}}(\psi_{00} + \epsilon(T\psi_{10} + \psi_{01}) + O(\epsilon^2)), \\ \Omega &= \Omega_{00} + \epsilon(T\Omega_{10} + \Omega_{01}) + O(\epsilon^2).\end{aligned}\right\} \tag{3.6}$$

The right-hand sides of (3.1) and (3.2) are then $O(T)$ and max $\{O(T), O(1)\}$ respectively. We shall henceforth assume that T is $O(1)$ and that ϵ is the only small parameter (in other words R_M is $O(\epsilon^{\frac{1}{2}})$).

The system of partial differential equations (3.1) and (3.2) and boundary conditions (3.3) and (3.4) defines an eigenvalue problem

$$F(\sigma, T, \epsilon, k) = 0 \tag{3.7}$$

which must be solved for each value of θ. R_M does not appear explicitly in (3.7) because it is a function of T and ϵ. The flow is unstable if there exist solutions of (3.7) with $\mathrm{Re}\,\sigma > 0$.

Since the azimuthal velocity of the inner boundary decreases monotonically with latitude we may expect that the flow is most unstable at the equator ($\theta = \frac{1}{2}\pi$). A critical Taylor number T_c may then be defined as the lowest value of T which satisfies (3.3) with $\mathrm{Re}\,\sigma = 0$ and k real at $\theta = \frac{1}{2}\pi$. For $\epsilon = 0$ the problem reduces to that for concentric cylinders, for which the solution is well known, namely $T_c = 1694{\cdot}95$ with $k = 3{\cdot}1265$ and $\sigma \equiv 0$. We shall assume that the critical conditions are determined by $\sigma \equiv 0$ for $\epsilon > 0$ also.

At higher latitudes the flow is subcritical but we may nevertheless obtain solutions with $T = T_c$ and $\sigma \equiv 0$ by allowing k to take complex values. In fact, k bifurcates at $\theta = \frac{1}{2}\pi$ and gives rise to two complex values with $\mathrm{Re}\,k > 0$ for $\theta \neq \frac{1}{2}\pi$. One of these, which we shall denote by $k^{(1)}$, has negative imaginary part for $\theta < \frac{1}{2}\pi$ and therefore represents solutions which decay exponentially with latitude. The second solution, $k^{(2)}$, demonstrates exponential growth but is equally acceptable because the domain is finite. It turns out that both solutions are needed to satisfy the finiteness condition (3.4) on $\operatorname{cosec}\theta\,\partial\overline{\psi}/\partial\theta$ at $\theta = 0, \pi$.

We now substitute the expansions (3.6) and (3.5) of the basic flow, and the perturbation for ϵ small into the perturbation equations (3.1) and (3.2). A corresponding expansion

$$T = T_0 + \epsilon T_1 + \epsilon^2 T_2 + \ldots \tag{3.8}$$

is assumed for T, in which $T_0 = 1694{\cdot}95$ and T_1 will be determined.

It turns out that this form of solution is valid only in mid-latitudes; near the equator ($\theta = \frac{1}{2}\pi$) and the poles ($\theta = 0, \pi$) separate solutions must be obtained. These are discussed in §§ 5 and 6.

4. The linear stability problem in mid-latitudes

4.1. *Order* ϵ^0

On substituting (3.5), (3.6) and (3.8) into (3.1) and (3.2) and equating terms $O(\epsilon^0)$, we obtain

$$\left.\begin{aligned} Mg_0 + ikT_0 \sin\theta f_0 &= 0, \\ Nf_0 - Pg_0 &= 0, \end{aligned}\right\} \tag{4.1}$$

where

$$M \equiv (D^2 - k^2 - \sigma) - ikT_0 a'_{00}(x) \sin 2\theta,$$

$$N \equiv (D^2 - k^2)(D^2 - k^2 - \sigma) - ikT_0 \sin 2\theta (a'_{00}(x)(D^2 - k^2) - a'''_{00}(x)),$$

$$P \equiv 2\cos\theta((1-x)D - 1) - 2ik(1-x)\sin\theta,$$

$$D \equiv \partial/\partial x$$

and primes denote differentiation with respect to x. The boundary conditions and a normalization condition are

$$f_0 = f_0' = g_0 = 0 \quad \text{at} \quad x = 0, 1; \quad f_0''' = 1 \quad \text{at} \quad x = 0. \tag{4.2}$$

Here θ is effectively a parameter and the system (4.1) may be regarded as one of ordinary differential equations in x forming an eigenvalue problem for k in terms of σ and T_0 for each value of θ. For $\theta = \frac{1}{2}\pi$, these equations reduce to those describing the instability of the flow between infinite concentric cylinders, i.e. curvature of the boundaries is neglected in this approximation. This is the classic Taylor-vortex problem and the solution is well known: the critical value of the Taylor number is $T_0 = 1694{\cdot}95$ when $k = k_0 = 3{\cdot}1265$ with σ real and therefore equal to zero. We assume that σ does not vary with θ and may henceforth be taken to be zero everywhere.

Away from the equator the solution bifurcates and there are two complex values of k with $\operatorname{Re} k > 0$ for each value of θ. We denote the two eigenvalues by $k^{(1)}$ and $k^{(2)}$ and the corresponding eigenfunctions by $(f_0^{(1)}(\theta, x), g_0^{(1)}(\theta, x))$ and $(f_0^{(2)}(\theta, x), g_0^{(2)}(\theta, x))$. Numerical computation of k indicates that the imaginary part of each eigenvalue has constant sign in $(0, \frac{1}{2}\pi)$. We define $k^{(1)}$ and $k^{(2)}$ such that

$$\operatorname{Im} k^{(1)} < 0, \quad \operatorname{Im} k^{(2)} > 0 \quad \text{in} \quad (0, \tfrac{1}{2}\pi).$$

Then the corresponding solutions for $\overline{\psi}$ and $\overline{\Omega}$ decay or grow exponentially with a decrease in θ from $\frac{1}{2}\pi$. The range of θ is finite so both solutions are acceptable.

We may see from (4.1) that $k^{(3)} = -\tilde{k}^{(1)}$ and $k^{(4)} = -\tilde{k}^{(2)}$ are also eigenvalues (a tilde here denotes the complex conjugate). In addition there are two eigenvalues, which we shall denote by $k^{(5)}$ and $k^{(6)}$, that are purely imaginary at $\theta = \frac{1}{2}\pi$ and are complex conjugates of one another there. Computed results show that they remain purely imaginary for all values of θ.

From (4.1) it may also be noted that if $k^{(j)}(\theta)$ $(j = 1, \ldots, 6)$ is an eigenvalue then so is $\tilde{k}^{(j)}(\pi - \theta)$ $(j = 1, \ldots, 6)$, and therefore the equations need be solved only for a half-range, $(0, \frac{1}{2}\pi)$ say. There are two possible ways of continuing $k^{(1)}$, $k^{(2)}$ and the corresponding eigenfunctions into $(\frac{1}{2}\pi, \pi)$: either

$$k^{(1)}(\pi - \theta) = \tilde{k}^{(2)}(\theta) \quad \text{or} \quad k^{(1)}(\pi - \theta) = \tilde{k}^{(1)}(\theta).$$

Both solutions are continuous at $\theta = \frac{1}{2}\pi$ because $k^{(1)}(\frac{1}{2}\pi) = k^{(2)}(\frac{1}{2}\pi)$ and is a real number, but considerations of symmetry about $\theta = \frac{1}{2}\pi$ lead us to choose the latter. That such a choice satisfies the conditions at the poles and matches with the solution near the equator is demonstrated in §§ 5 and 6. Similarly,

$$k^{(5)}(\pi - \theta) = \tilde{k}^{(6)}(\theta) \quad \text{and} \quad k^{(6)}(\pi - \theta) = \tilde{k}^{(5)}(\theta).$$

The solutions with indices 1 and 3 then oscillate and decay rapidly with latitude on both sides of the equator while those with indices 2 and 4 grow. The solutions with indices 5 and 6 do not oscillate with latitude and simply grow or decay exponentially. The general form of solution is therefore

$$(\overline{\psi}, \overline{\Omega}) = \sum_{j=1}^{6} A_0^{(j)}(\theta) \exp\left\{\frac{1}{\epsilon}\int_{\frac{1}{2}\pi}^{\theta} k^{(j)}\, d\theta\right\} ((\epsilon T)^{\frac{1}{2}} f_0^{(j)}(\theta, x), g_0^{(j)}(\theta, x)) + O(\epsilon), \tag{4.3}$$

but we shall show in § 6 that the coefficients of the two modes with indices 5 and 6 are zero at least to leading order in ϵ.

At this stage each mode may be discussed separately and we temporarily drop the index (j).

4.2. *Adjoint equations*

Later on we shall need to use the system of ordinary differential equations adjoint to the system (4.1) for $\sigma = 0$. This is

$$\left.\begin{aligned} M^{\dagger}g_0^{\dagger} - P^{\dagger}f_0^{\dagger} &= 0, \\ N^{\dagger}f_0^{\dagger} - ikT_0 \sin\theta g_0^{\dagger} &= 0, \end{aligned}\right\} \tag{4.4}$$

where

$$M^{\dagger} \equiv (D^2 - k^2) - ikT_0 a'_{00}(x) \sin 2\theta,$$

$$N^{\dagger} \equiv (D^2 - k^2)^2 - ikT_0 \sin 2\theta [a'_{00}(x)(D^2 - k^2) + 2a''_{00}(x) D],$$

$$P^{\dagger} \equiv 2\cos\theta(1-x) D + 2ik(1-x)\sin\theta.$$

The boundary conditions are

$$f_0^{\dagger} = f_0^{\dagger\prime} = g_0^{\dagger} = 0 \quad \text{at} \quad x = 0, 1; \quad f_0^{\dagger\prime\prime\prime} = 1 \quad \text{at} \quad x = 0.$$

The eigenvalues k and T_0 are the same as those of the system (4.1) but of course the adjoint function pair $(f_0^{\dagger}, g_0^{\dagger})$ is different from (f_0, g_0).

4.3. *Order* ϵ

Equating terms in (3.1) and (3.2) in ϵ, we obtain

$$A_1(Mg_1 + ikT_0 \sin\theta f_1) = (G_{11} + T_1 G_{12} + ik\cot\theta g_0) A_0 + (G_{13} - 2ikg_0)\, dA_0/d\theta, \tag{4.5a}$$

$$\begin{aligned} A_1(Nf_1 - Pg_1) = {} & (F_{11} + T_1 F_{12} + 2ik\cot\theta(D^2 - k^2) f_0) A_0 \\ & + (F_{13} - 4ik(D^2 - k^2) f_0)\, dA_0/d\theta, \end{aligned} \tag{4.5b}$$

with

$$f_1 = f_1' = g_1 = 0 \quad \text{at} \quad x = 0, 1; \quad f_1''' = 1 \quad \text{at} \quad x = 0. \tag{4.6}$$

Here

$$\begin{aligned} F_{11} = {} & -4xk^2(D^2 - k^2) f_0 - 2x\{2\cos\theta[(1-x) D - 1] - 3ik(1-x)\sin\theta\} g_0 \\ & - 2ikxT_0 \sin 2\theta [a'_{00}(x)(D^2 - k^2) - a'''_{00}(x)]\, f_0 \\ & + ik^2 a'_{00}(x) T_0 \sin 2\theta (2xk - i\cot\theta) f_0 - 4(1-x)\cos\theta\, g_0 \\ & + 2(3\cos^2\theta - 1) T_0 [a''_{00}(x) - a_{00}(D^2 - k^2)] Df_0 \\ & - 4T_0 \cos\theta\{\cos\theta[a'''_{00}(x) D + a'_{00}(D^2 - k^2)] - ika''_{00}(x)\sin\theta\}\, f_0 \\ & - (dk/d\theta)[2i(D^2 - 3k^2) + 3kT_0 \sin 2\theta a'_{00}(x)] f_0 + 2(1-x)\sin\theta\, \partial g_0/\partial\theta \\ & + T_0 \sin 2\theta [a'_{00}(x)(D^2 - 3k^2) - a'''_{00}(x)]\, \partial f_0/\partial\theta - 4ik(D^2 - k^2)\, \partial f_0/\partial\theta \\ & + 2\operatorname{cosec}^2\theta[\cos\theta(\Omega_1 D + \Omega_{1x}) - ik\sin\theta\Omega_1] g_0 - ikT_0(\psi_{1x}(D^2 - k^2) - \psi_{1xxx}) f_0, \end{aligned}$$

$$F_{12} = ik\sin 2\theta [a'_{00}(x)(D^2 - k^2) - a'''_{00}(x)] f_0,$$

$$F_{13} = -2(1-x)\sin\theta g_0 + T_0 \sin 2\theta (a'_{00}(x)(D^2 - 3k^2) - a'''_{00}(x)) f_0,$$

$$\begin{aligned} G_{11} = {} & (2ikx + 2xa'_{00}(x) T_0 \sin 2\theta + \operatorname{cosec}\theta T_0 \psi_{1x})\, ikg_0 \\ & + (2xT_0 \sin\theta + \operatorname{cosec}\theta\Omega_{1x})\, ikf_0 - 2T_0 a_{00}(x)(3\cos^2\theta - 1) Dg_0 \\ & - 2T_0(1-x)\cos\theta Df_0 - ig_0 dk/d\theta - 2ik\, \partial g_0/\partial\theta - T_0 \sin\theta\, \partial f_0/\partial\theta \\ & + T_0 \sin 2\theta a'_{00}(x)\, \partial g_0/\partial\theta, \end{aligned}$$

$$G_{12} = (D^2 - k^2) g_0/T_0,$$

$$G_{13} = (D^2 - k^2) g_0/ik = -T_0 \sin\theta f_0 + T_0 a_{00}(x) \sin 2\theta g_0,$$

and ψ_1 and Ω_1 denote $(\psi_{01} + T_0\psi_{10})$ and $(\Omega_{01} + T_0\Omega_{10})$ respectively.

The operators on the left of (4.5) are the same as those on the left of (4.1) and the terms on the right contain f_0 and g_0, which are known once (4.1) has been solved, and T_1, which is yet to be determined. The amplitude functions $A_0(\theta)$ and $A_1(\theta)$ are as yet unknown; $A_0(\theta)$ may be determined as follows, but it is necessary to examine terms $O(\epsilon^2)$ to calculate $A_1(\theta)$.

Since a non-trivial solution to the homogeneous equations (4.1) exists, the inhomogeneous equations (4.5) have a solution if and only if their right-hand sides are orthogonal to the adjoint solution of (4.1), i.e. $(f_0^\dagger, g_0^\dagger)$. The orthogonality condition requires that multiplying the right-hand sides of (4.5a, b) by $g_0^\dagger$ and $f_0^\dagger$ respectively, integrating with respect to x over (0, 1) and subtracting yields zero. This gives the following relation between A_0, $dA_0/d\theta$, T_0, T_1, k and θ:

$$(H_{11}+T_1H_{12}+h_1)\,A_0(\theta)+(H_{13}+h_3)\,dA_0/d\theta = 0, \tag{4.7}$$

where
$$H_{1i} = \int_0^1 (g_0^\dagger G_{1i} - f_0^\dagger F_{1i})\,dx \quad (i = 1, 2, 3),$$

$$h_1 = ik\cot\theta \int_0^1 [g_0 g_0^\dagger - f_0^\dagger\, 2(D^2-k^2) f_0]\,dx,$$

$$h_3 = -2h_1 \tan\theta.$$

Using the definitions of F_{1i} and G_{1i} ($i = 1, 2, 3$) given above, we see that as $\theta \to 0$

$$dA_0/d\theta \sim \tfrac{1}{2}\cot\theta A_0, \quad \text{or} \quad A_0 \sim (\sin\theta)^{\frac{1}{2}}.$$

The singularities in $dA_0/d\theta$ at $\theta = 0, \pi$ may be brought out by writing

$$A_0(\theta) = (\sin\theta)^{\frac{1}{2}}\,\bar{A}_0(\theta).$$

Then (4.7) becomes

$$(H_{11}+\tfrac{1}{2}\cot\theta\, H_{13}+T_1 H_{12})\,\bar{A}_0(\theta)+(H_{13}+h_3)\,d\bar{A}_0/d\theta = 0. \tag{4.8}$$

The singularity still exists, of course, and we return to it in § 6.

In the next section we demonstrate that, as $\theta \to \frac{1}{2}\pi$, $H_{13}+h_3 \to 0$ for $j = 1, \ldots, 4$ while the coefficient of $\bar{A}_0(\theta)$ in (4.8) remains non-zero. Near $\theta = \frac{1}{2}\pi$ a new scaling is needed but elsewhere (4.8) may be integrated to give

$$\bar{A}_0(\theta) = -\alpha \exp \int_0^\theta \chi(\theta)\,d\theta, \tag{4.9}$$

where $\chi(\theta) = (H_{11}+\frac{1}{2}\cot\theta + H_{13}+T_1H_{12})/(H_{13}+h_3)$ and α is a constant of integration. The relative magnitudes of the six coefficients α_j ($j = 1, \ldots, 6$) are determined by conditions at the poles and the equator but of course their absolute magnitudes remain arbitrary in a linear theory. Details of the calculations are given in §§ 5 and 6.

5. The neighbourhood of the equator

At the equator ($\theta = \frac{1}{2}\pi$), the eigenvalue problem for k reduces to that for Taylor instability between infinite concentric cylinders with a small gap-to-radius ratio. It is well known that for $T \geqslant T_0 = 1694{\cdot}95$ there are four real solutions for k and two purely imaginary ones. At $T = T_0$ the four real ones become $\pm k_0 (= \pm 3{\cdot}1265)$, twice, and therefore $k^{(1)} = k^{(2)}$ and $k^{(3)} = k^{(4)}$. It is this coincidence of eigenvalues that causes the difficulties at the equator.

In order to discuss the solution near the equator in more detail we first investigate the limit of the mid-latitude solution as $\theta \to \frac{1}{2}\pi$. The eigenvalues $k^{(5)}$ and $k^{(6)}$ present no special difficulty and $k^{(3)}$ and $k^{(4)}$ may be written in terms of $k^{(1)}$ and $k^{(2)}$, so that we may confine our attention for now to $k^{(j)}$ with $j = 1, 2$ and again drop the index. Let us write $\theta = \frac{1}{2}\pi - \phi$ and suppose first of all that k, f_0 and g_0 have regular expansions in ϕ for $|\phi| \ll 1$, i.e.

$$k = k_0 + \phi k_1 + \phi^2 k_2 + \dots,$$

$$(f_0, g_0) = (f_{00}, g_{00}) + \phi(f_{01}, g_{01}) + \phi^2(f_{02}, g_{02}) + \dots.$$

Then substituting into (4.1) and equating terms $O(\phi^0)$ gives the classical Taylor problem

$$\left.\begin{aligned} M_0 g_{00} + ik_0 T_0 f_{00} &= 0,\\ M_0^2 f_{00} + 2ik_0(1-x) g_{00} &= 0,\end{aligned}\right\} \tag{5.1}$$

where $M_0 \equiv D^2 - k_0^2$. Terms $O(\phi)$ give

$$\left.\begin{aligned} M_0 g_{01} + ik_0 T_0 f_{01} &= -ik_1 T_0 f_{00} + 2k_0 k_1 g_{00} + 2ik_0 T_0 a'_{00} g_0,\\ M_0^2 f_{01} + 2ik_0(1-x) g_{01} &= -2i(1-x) k_1 g_{00} + 4k_0 k_1 M_0 f_{00} + 2((1-x) g_{00})'\\ &\quad + 2ik_0 T_0(a'_{00} M_0 - a'''_{00}) f_{00}.\end{aligned}\right\} \tag{5.2}$$

The orthogonality condition for a solution of (5.2) to exist is

$$k_1 = A/B,$$

where

$$\left.\begin{aligned} A &= \int_0^1 \{2ik_0 T_0 a'_{00} g_{00} g^\dagger_{00} - 2ik_0 T_0(a'_{00} M_0 - a'''_0) f_{00} f^\dagger_{00} - 2[(1-x) g_{00}]' f^\dagger_{00}\}\, dx,\\ B &= \int_0^1 \{(iT_0 f_{00} - 2k_0 g_{00}) g^\dagger_{00} - [2i(1-x) g_{00} - 4k_0 M_0 f_{00}] f^\dagger_{00}\}\, dx\end{aligned}\right\} \tag{5.3}$$

and $(f^\dagger_{00}, g^\dagger_{00})$ is the adjoint pair at $\theta = \frac{1}{2}\pi$.

Now the condition that T_0 is a minimum as a function of k_0 with k_0 real may be obtained by differentiating (5.1) with respect to k_0 and setting $\partial T_0/\partial k_0 = 0$. A solution of the inhomogeneous equations for $\partial g_{00}/\partial k_0$ and $\partial f_{00}/\partial k_0$ exists provided that

$$\int_0^1 \{(iT_0 f_{00} - 2k_0 g_{00}) g^\dagger_{00} - [2i(1-x) g_{00} - 4k_0 M_0 f_{00}] f^\dagger_{00}\}\, dx = 0. \tag{5.4}$$

This means that the denominator of (5.3) vanishes and indicates that the expansions about k_0, f_{00} and g_{00} are not regular. An appropriate expansion is found to proceed in powers of $\phi^{\frac{1}{2}}$, i.e.

$$\left.\begin{aligned} k &= k_0 + \phi^{\frac{1}{2}} k_1 + \phi k_2 + \dots,\\ (f_0, g_0) &= (f_{00}, g_{00}) + \phi^{\frac{1}{2}} k_1(f_{01}, g_{01}) + \phi[(f_{02}, g_{02}) + k_2(f_{01}, g_{01})] + \dots.\end{aligned}\right\} \tag{5.5}$$

Then f_{00} and g_{00} again satisfy (5.1), but f_{01} and g_{01} now satisfy

$$\left.\begin{aligned} M_0 g_{01} + ik_0 T_0 f_{01} &= -iT_0 f_{00} + 2k_0 g_{00},\\ M_0^2 f_{01} + 2ik_0(1-x) g_{01} &= -2i(1-x) g_{00} + 4k_0 M_0 f_{00}.\end{aligned}\right\} \tag{5.6}$$

The condition that a solution of (5.6) exists is equivalent to (5.4).

Terms $O(\phi)$ now give

$$\left.\begin{aligned} M_0 g_{02} + ik_0 T_0 f_{02} &= k_1^2(-iT_0 f_{01} + 2k_0 g_{01} + g_{00}) + 2ik_0 T_0 a'_{00} g_{00}, \\ M_0^2 f_{02} + 2ik_0(1-x) g_{02} &= 2k_1^2[(M_0 - 2k_0^2) f_{00} + 2k_0 M_0 f_{01} - i(1-x) g_{01}] \\ &\quad + 2ik_0 T_0(a'_{00} M_0 - a_0''') f_{00} + 2((1-x) g_{00})'. \end{aligned}\right\} \tag{5.7}$$

The orthogonality condition for a solution of (5.7) to exist yields an equation for k_1^2:

$$k_1^2 = -A/B,$$

where

$$\left.\begin{aligned} A &= \int_0^1 \{2ik_0 T_0 a'_{00} g_{00} g_{00}^\dagger - 2[(1-x) g'_{00}] f_{00}^\dagger - 2ik_0 T_0(a'_{00} M_0 - a_{00}''') f_{000} f_{00}^\dagger\}\, dx, \\ B &= \int_0^1 [(g_{00} + 2k_0 g_{01} - iT_0 f_{01}) g_{00}^\dagger - 2(M_0 - 2k_0^2) f_{00} f_{00}^\dagger \\ &\qquad - 4k_0 M_0 f_{01} f_{00}^\dagger + 2i(1-x) g_{01} f_{00}^\dagger]\, dx. \end{aligned}\right\} \tag{5.8}$$

It will be observed that, since f_{00}, f_{01} and $f_{00}^\dagger$ are real and g_{00}, g_{01} and $g_{00}^\dagger$ pure imaginary, we may write $k_1^2 = -iK^2$, where K^2 is real; the computed value of K^2 is 4·0199. We denote the solutions of (5.8) by

$$k_1^{(1)} = e^{-\frac{1}{4}\pi i} K, \quad k_1^{(2)} = e^{\frac{3}{4}\pi i} K.$$

We now turn our attention to the behaviour of the amplitude $\bar{A}_0(\theta)$, which satisfies (4.8) as $\theta \to \frac{1}{2}\pi$. Using the expansion (5.5), it follows from the definitions of H_{11}, H_{12}, H_{13} and h_3 in (4.7) that as $\phi \to 0$

$$\begin{aligned} H_{11} &\sim -\tfrac{1}{2} k_1 \phi^{-\frac{1}{2}} \int_0^1 \{(g_{00} + 2k_0 g_{01} - iT_0 f_{01}) g_{00}^\dagger - [2(M_0 - 2k_0^2) f_{00} \\ &\qquad + 4k_0 M_0 f_{01} - 2i(1-x) g_{01}] f_{00}^\dagger\}\, dx, \\ H_{12} &\sim -ik_0 T_0 \int_0^1 g_{00} f_{00}^\dagger\, dx, \\ h_3 + H_{13} &\sim i \int_0^1 \{(iT_0 f_{00} - 2k_0 g_{00}) g_{00}^\dagger - [2i(1-x) g_{00} - 4k_0 M_0 f_{00}) f_{00}^\dagger\}\, dx \\ &\quad + 2k_1 \phi^{\frac{1}{2}} \int_0^1 \{(g_{00} + 2k_0 g_{01} - iT_0 f_{01}) g_{00}^\dagger - [2(M_0 - 2k_0^2) f_{00} \\ &\qquad + 4k_0 M_0 f_{01} - 2i(1-x) g_{01}] f_{00}^\dagger\}\, dx. \end{aligned}$$

Furthermore, h_3 remains $O(1)$ and $H_{13} \cot\theta$ is proportional to ϕ as $\phi \to 0$. The leading term in $h_3 + H_{13}$ vanishes by virtue of (5.4) and it follows that the coefficient of $d\bar{A}_0/d\theta$ in (4.8) vanishes at $\bar{\phi} = 0$. Asymptotically (4.8) becomes

$$2(\phi^{\frac{1}{2}} + \mu_1 \phi + \ldots) d\bar{A}_0/d\phi + \tfrac{1}{2}(\phi^{-\frac{1}{2}} + \mu_0 + \ldots) \bar{A}_0 = 0,$$

where μ_0 and μ_1 are constants given in terms of integrals involving $f_{00}, f_{01}, f_{00}^\dagger$, etc. It follows that

$$\bar{A}_0(\phi) \sim \alpha' \phi^{-\frac{1}{4}} \exp\{(\mu_1 - \mu_0) \phi^{\frac{1}{2}}\} \quad \text{as} \quad \phi \to 0,$$

where α' is a constant.

It is convenient to define that part of χ whose integral is regular at $\phi = 0$ by χ^* and let

$$\chi = -\tfrac{1}{4}\phi + \chi^*.$$

Then
$$\bar{A}_0(\phi) = \alpha\left(\frac{\pi}{2\phi}\right)^{\frac{1}{4}} \exp\int_{\frac{1}{2}\pi}^{\phi} \chi^* \, d\phi$$
$$\sim \alpha\left(\frac{\pi}{2\phi}\right)^{\frac{1}{4}} \exp\left(\int_{\frac{1}{2}\pi}^{0} \chi^* \, d\phi + \tfrac{1}{4}(\mu_1-\mu_0)\,\phi^{\frac{1}{2}}\right) \quad \text{as} \quad \phi \to 0.$$

Using the expansion (5.5) for k and the definition (4.3) of $\bar{\psi}$, we obtain

$$\bar{\psi} \sim (\epsilon T)^{\frac{1}{2}} f_{00}(x)\left(\frac{\pi}{2\phi}\right)^{\frac{1}{4}} \sum_{j=1}^{4} \alpha_j \exp\left\{-i\frac{k_0^{(j)}}{\epsilon}\phi - \frac{2i}{\epsilon}k_1^{(j)}\phi^{\frac{3}{2}} + \dots + \int_{\frac{1}{2}\pi}^{0} \chi^{*(j)} d\phi + \tfrac{1}{4}(\mu_1^{(j)} - \mu_0^{(j)})\,\phi^{\frac{1}{2}} + \dots\right\} \quad \text{as} \quad \phi \to 0, \tag{5.9}$$

and similarly for $\bar{\Omega}$. Here $k_0^{(1)} = k_0^{(2)} = -k_0^{(3)} = -k_0^{(4)}$, $k_1^{(1)} = k_1^{(3)} = e^{-\frac{1}{4}\pi i}K$ and $k_1^{(2)} = k_1^{(4)} = e^{\frac{3}{4}\pi i}K$. It is only the four eigenvalues $k^{(j)}$ with $j = 1, \dots, 4$ that possess the square-root behaviour in ϕ as $\phi \to 0$ and consequently only these that make H_{11} singular and $h_3 + H_{13}$ zero there. The modes associated with $j = 5$ and 6 do not behave like $\phi^{-\frac{1}{4}}$ as $\phi \to 0$ and therefore do not appear at leading order in (5.9). We note, however, that their contribution to $\bar{\psi}$ satisfies the condition of antisymmetry $\bar{\psi}(-\phi) = -\bar{\psi}(\phi)$ only if

$$\alpha_5 \exp\left\{\int_{\frac{1}{2}\pi}^{0} \chi^{*(5)} d\phi\right\} = -\alpha_6 \exp\left\{\int_{\frac{1}{2}\pi}^{0} \chi^{*(6)} d\phi\right\}. \tag{5.10}$$

$\bar{\psi}$ and $\bar{\Omega}$ are therefore singular at $\phi = 0$ and a new scaling is needed when ϕ is small. The expansion (5.9) in powers of ϵ breaks down when any of the terms in the exponent is $O(1)$ for then it is comparable to the algebraic term. For small ϕ the third and subsequent terms are small and the second and first terms are $O(1)$ when $\phi = O(\epsilon^{\frac{2}{3}})$ and $O(\epsilon)$ respectively. We shall see later that a scaling $\phi \sim \epsilon$ is necessary, but first we examine $\phi \sim \epsilon^{\frac{2}{3}}$.

Let us write $\phi = \epsilon^{\frac{2}{3}}\,\bar{\phi}$ with $\bar{\phi} = O(1)$ and then expand $\bar{\psi}$ in the form

$$\bar{\psi} = (\epsilon T)^{\frac{1}{2}} \exp\{-ik_0\,\epsilon^{-\frac{1}{3}}\bar{\phi}\}\,[B_0(\bar{\phi})f_{00}(x) + \epsilon^{\frac{1}{3}}B_1(\bar{\phi})\bar{f}_{01}(x) + \epsilon^{\frac{2}{3}}B_2(\bar{\phi})\bar{f}_{02}(x) + \dots] + \text{c.c.},$$

where c.c. denotes the complex conjugate, and similarly for $\bar{\Omega}$. Here $B_0(\bar{\phi}_1)$, $B_1(\bar{\phi})$, etc. are functions of $\bar{\phi}$ to be determined and $\bar{f}_{01}, \bar{f}_{02}$, etc. are functions of x which will be shown to be related to f_{01}, f_{02}, etc. On substituting this form of solution into (3.1) and (3.2) and equating coefficients of ϵ^0 we obtain (5.1) again, and equating coefficients of $\epsilon^{\frac{1}{3}}$ we obtain

$$B_1(\bar{\phi})\bar{f}_{01}(x) = \frac{idB_0}{d\bar{\phi}}f_{01}(x) + C_0(\bar{\phi})f_{00}(x), \tag{5.11}$$

where $f_{01}(x)$ is given by (5.6) and $C_0(\bar{\phi})$ is an arbitrary function.

The terms $O(\epsilon^{\frac{2}{3}})$ satisfy

$$\left.\begin{aligned} B_2(M_0\bar{g}_{02} + ik_0T_0\bar{f}_{02}) &= \frac{-d^2B_0}{d\bar{\phi}^2}g_{00} + 2ik_0\bar{g}_{01}\frac{dB_1}{d\bar{\phi}} + T_0\bar{f}_{01}\frac{dB_1}{d\bar{\phi}} + 2ik_0T_0a'_{00}g_{00}B_0\bar{\phi}, \\ B_2(M_0^2\bar{f}_{02} + 2ik_0(1-x)\,\bar{g}_{02}) &= -(2M_0 - 4k_0^2)f_{00}\frac{d^2B_0}{d\bar{\phi}^2} + (4ik_0M_0\bar{f}_{01} + 2(1-x)\,\bar{g}_{01})\frac{dB_1}{d\bar{\phi}} \\ &\quad + [2ik_0T_0\bar{\phi}(a'_{00}M_0^2 - a'''_{00})f_{00} + 2\bar{\phi}(1-x)\,g_{00})']\,B_0. \end{aligned}\right\} \tag{5.12}$$

We may then write

$$B_2 \bar{f}_{02} = \bar{\phi} B_0 f_{02} + \frac{idC_0}{d\bar{\phi}} f_{01} + C_1(\bar{\phi}) f_{00}, \tag{5.13}$$

and similarly for $B_2 \bar{g}_{02}$, where $C_1(\bar{\phi})$ is an arbitrary function of $\bar{\phi}$ and f_{02} satisfies (5.7). The orthogonality condition then gives

$$d^2B_0/d\bar{\phi}^2 + k_1^2 \bar{\phi} B_0 = 0. \tag{5.14}$$

Let us write $k_1^2 = e^{-\frac{1}{2}i\pi} K^2$ with K real; then (5.14) becomes

$$d^2B_0/d\bar{\phi}^2 - e^{-\frac{3}{2}\pi i} K^2 \bar{\phi} B_0 = 0.$$

The solution may be written as

$$B_0(\bar{\phi}) = \beta_1(\mathrm{Bi}\,(z) - \mathrm{Ai}\,(z)) + \beta_2 \,\mathrm{Ai}\,(z), \quad z = e^{-\frac{1}{2}i\pi} K^{\frac{2}{3}} \bar{\phi}, \tag{5.15}$$

where $\mathrm{Ai}\,(z)$ and $\mathrm{Bi}\,(z)$ are Airy functions and β_1 and β_2 are arbitrary constants.

To leading order in ϵ we then have

$$\bar{\psi}(\bar{\phi}) = \exp\{-i\epsilon^{-\frac{1}{3}} k_0 \bar{\phi}\} B_0(\bar{\phi}) f_{00}(x) + \exp\{i\epsilon^{-\frac{1}{3}} k_0 \bar{\phi}\} \tilde{B}_0(\bar{\phi}) f_{00}(x) + \ldots, \tag{5.16}$$

where $f_{00}(x)$ is a real function of x. We require $\bar{\psi}(\bar{\phi})$ to be antisymmetric about $\bar{\phi} = 0$, i.e. $\bar{\psi}(-\bar{\phi}) = -\bar{\psi}(\bar{\phi})$. It follows from (5.14) that

$$B_0(-\bar{\phi}) = -\tilde{B}_0(\bar{\phi}) \tag{5.17}$$

and from (5.15) that
$$\beta_1 + \tilde{\beta}_1 = \beta_2 + \tilde{\beta}_2 = 0.$$

Equivalent conditions on α_1 and α_2 are obtained by matching with the solution in higher latitudes.

Now, as $\bar{\phi} \to \infty$

$$\begin{aligned}
\mathrm{Ai}\,(z) &\sim (4\pi)^{-\frac{1}{2}} z^{-\frac{1}{4}} \exp\{-\tfrac{2}{3} z^{\frac{3}{2}}\} \\
&= (4\pi)^{-\frac{1}{2}} e^{\frac{1}{8}\pi i} K^{-\frac{1}{6}} \bar{\phi}^{-\frac{1}{4}} \exp\{-\tfrac{2}{3} e^{-\frac{3}{4}\pi i} K \bar{\phi}^{\frac{3}{2}}\} \\
&= (4\pi)^{-\frac{1}{2}} e^{\frac{1}{8}\pi i} K^{-\frac{1}{6}} \bar{\phi}^{-\frac{1}{4}} \exp\{-\tfrac{2}{3} i k_1^{(2)} \bar{\phi}^{\frac{3}{2}}\},
\end{aligned}$$

$$\begin{aligned}
\mathrm{Bi}\,(z) &= \mathrm{Bi}\,(e^{-\frac{1}{3}\pi i} z_1), \quad \text{where} \quad z_1 = e^{-\frac{1}{6}\pi i} K^{\frac{2}{3}} \bar{\phi}, \\
&\sim (\tfrac{1}{2}\pi)^{-\frac{1}{2}} e^{-\frac{1}{6}\pi i} z_1^{-\frac{1}{4}} \sin(\tfrac{2}{3} z_1^{\frac{3}{2}} + \tfrac{1}{4}\pi + i \ln 2^{\frac{1}{2}}) \\
&= (4\pi)^{-\frac{1}{2}} e^{\frac{1}{8}\pi i} K^{-\frac{1}{6}} \bar{\phi}^{-\frac{1}{4}} [\exp\{-\tfrac{2}{3} i e^{\frac{3}{4}\pi i} K \bar{\phi}^{\frac{3}{2}}\} + 2 e^{\frac{1}{2}\pi i} \exp\{-\tfrac{2}{3} i e^{-\frac{1}{4}\pi i} K \bar{\phi}^{\frac{3}{2}}\}]
\end{aligned}$$

and
$$\mathrm{Bi}\,(z) - \mathrm{Ai}\,(z) \sim 2(4\pi)^{-\frac{1}{2}} e^{\frac{5}{8}\pi i} K^{-\frac{1}{6}} \bar{\phi}^{-\frac{1}{4}} \exp\{-\tfrac{2}{3} i k_1^{(1)} \bar{\phi}^{\frac{3}{2}}\}.$$

Hence
$$B_0(\bar{\phi}) \sim (4\pi)^{-\frac{1}{2}} e^{\frac{1}{8}\pi i} K^{-\frac{1}{6}} \bar{\phi}^{-\frac{1}{4}} [2\beta_1 \exp\{-\tfrac{2}{3} i k_1^{(1)} \bar{\phi}^{\frac{3}{2}}\} + \beta_2 \exp\{-\tfrac{2}{3} i k_1^{(2)} \bar{\phi}^{\frac{3}{2}}\}]. \tag{5.18}$$

The solution given by (5.15)–(5.18) is equivalent to that given in (5.9) for the limit of the mid-latitude solution as $\phi \to 0$ if

$$\left.\begin{aligned}
\alpha_1 &= 2(\tfrac{1}{2}\pi)^{\frac{3}{4}} (\epsilon/K)^{\frac{1}{6}} e^{\frac{5}{8}\pi i} \exp\left\{-\int_0^{\frac{1}{2}\pi} \chi^{*(1)} \, d\phi\right\} \beta_1, \\
\alpha_2 &= (\tfrac{1}{2}\pi)^{\frac{3}{4}} (\epsilon/K)^{\frac{1}{6}} e^{\frac{1}{8}\pi i} \exp\left\{-\int_0^{\frac{1}{2}\pi} \chi^{*(2)} \, d\phi\right\} \beta_2.
\end{aligned}\right\} \tag{5.19}$$

The symmetry condition (5.17) then gives

$$\mathrm{Re}\left\{\alpha_1 e^{-\frac{3}{8}\pi i}\exp\left\{\int_0^{\frac{1}{2}\pi}\chi^{*(1)}\,d\phi\right\}\right\} = 0,$$

$$\mathrm{Re}\left\{\alpha_2 e^{-\frac{1}{8}\pi i}\exp\left\{\int_0^{\frac{1}{2}\pi}\chi^{*(2)}\,d\phi\right\}\right\} = 0,$$

or, equivalently,

$$\left.\begin{aligned}\alpha_1 &= \alpha_1' e^{\frac{3}{8}\pi i}\exp\left\{-\int_0^{\frac{1}{2}\pi}\chi^{*(1)}\,d\phi\right\},\\ \alpha_2 &= \alpha_2' e^{-\frac{3}{8}\pi i}\exp\left\{-\int_0^{\frac{1}{2}\pi}\chi^{*(2)}\,d\phi\right\},\end{aligned}\right\} \tag{5.20}$$

where α_1' and α_2' are real numbers.

At this stage the two modes $j = 1, 2$ may still be discussed separately and it would seem to be possible to reject the one which grows with latitude (index 2) by setting $\alpha_2 = \beta_2 = 0$. We shall see in § 6, however, that both solutions are needed in order to satisfy the boundary conditions at the pole $\theta = 0$. First, though, we must continue with the solution near the equator.

So far we have obtained the solution as far as the term in $\epsilon^{\frac{3}{4}}$ except for an arbitrary function $C_0(\overline{\phi})$ of $\overline{\phi}$ in (5.9). To this order the singularity in $\overline{\psi}$ and $\overline{\Omega}$ at the equator is smoothed out. If we now continue the expansion and examine terms in ϵ we can compute $C_0(\overline{\phi})$.

The terms $O(\epsilon)$ give

$$\begin{aligned}B_3(M_0\bar{g}_{03}+ik_0T_0\bar{f}_{03}) = {} & (2ik_0\bar{g}_{02}+T_0\bar{f}_{02})\,dB_2/d\phi-\bar{g}_{01}\,d^2B_1/d\overline{\phi}^2\\ & +2ik_0T_0a_{00}'\overline{\phi}\bar{g}_{01}B_1-2a_{00}'T_0\overline{\phi}g_{00}\,dB_0/d\overline{\phi}\\ & +ik_0T_0\Omega_{1x}f_{00}+2a_{00}T_0g_{00}'-2xk_0^2g_{00}+2ik_0T_0xf_{00}\\ & -ik_0T_1f_{00})\,B_0,\end{aligned} \tag{5.21a}$$

$$\begin{aligned}B_3(M_0^2\bar{f}_{03}+2ik_0(1-x)\,\bar{g}_{03} = {} & [4ik_0M_0\bar{f}_{02}+2(1-x)\bar{g}_{02}]\,dB_2/d\overline{\phi}-2(M_0-2k_0^2)\bar{f}_{01}\,d^2B_1/d\overline{\phi}^2\\ & +2\{ik_0T_0(a_{00}'M_0-a_{00}''')\bar{f}_{01}+[(1-x)\,\bar{g}_{01})'\}\,\overline{\phi}B_1\\ & -2T_0(a_{00}'M_0-a_{00}''')f_{00}\,\overline{\phi}\,dB_0/d\overline{\phi}\\ & +[6ik_0x(1-x)\,g_{00}-4xk_0^2M_0f_{00}-2ik_0\,\Omega_1g_{00}\\ & +2T_0(a_{00}'M_0-a_{00}''')f_{00}']\,B_0.\end{aligned} \tag{5.21b}$$

The orthogonality condition reduces to

$$\gamma_1\,d^3B_0/d\overline{\phi}^3+(\gamma_2+T_1\gamma_3)\,B_0 = \gamma_4(d^2C_0/d\overline{\phi}^2+\overline{\phi}k_1^2C_0), \tag{5.22}$$

where

$$\begin{aligned}\gamma_1 = \int_0^1 & \{k_1^{-2}(-2ik_0g_{02}-T_0f_{02}+2k_0T_0a_{00}'g_{01}-2a_{00}'T_0g_{00})\,g_{00}^\dagger+k_1^{-2}[4ik_0M_0f_{02}\\ & +2(1-x)\,g_{02}-2k_0T_0(a_{00}'M_0-a_{00}''')f_{01}+2i((1-x)\,g_{01})'\\ & -2T_0(a_{00}'M_0-a_{00}''')\,f_{00}]\,f_{00}^\dagger-ig_{01}g_{00}^\dagger+2i(M_0-2k_0^2)\,f_{01}f_{00}^\dagger-4ik_0f_{00}f_{00}^\dagger\}\,dx,\end{aligned}$$

$$\begin{aligned}\gamma_2 = \int_0^1 & \{2k_0T_0a_{00}'g_{01}g_{00}^\dagger-2k_0T_0(a_{00}'M_0-a_{00}''')f_{01}f_{00}^\dagger+ik_0T_0\Omega_{1x}f_{00}g_{00}^\dagger\\ & +2i[(1-x)\,g_{01}]'f_{00}^\dagger-2T_0(a_{00}'M_0-a_{00}''')f_{00}f_{00}^\dagger\\ & +(2a_{00}T_0g_{00}'-2k_0^2xg_{00}+2ik_0T_0xf_{00}-2a_{00}'T_0g_{00})\,g_{00}^\dagger\\ & -[6ik_0x(1-x)\,g_{00}-4xk_0^2M_0f_{00}-2ik_0\Omega_1g_{00}+2T_0(a_{00}M_0-a_0''')f_{00}']\,f_{00}^\dagger\}\,dx,\end{aligned}$$

$$\gamma_3 = -\int_0^1 ik_0 f_{00} g_{00}^\dagger \, dx,$$

$$\gamma_4 = -\int_0^1 \{(-g_{00} + iT_0 f_{01} - 2k_0 g_{01}) g_{00}^\dagger + [4k_0 M_0 f_{01} - 2i(1-x) g_{01} - 2i(M_0 - 2k_0^2) f_{00}] f_{00}^\dagger\} \, dx.$$

We have made use here of (5.13) and its derivative. A solution of (5.22) is

$$C_0(\overline{\phi}) = -\tfrac{1}{4}\gamma_1 k_1^2 B_0(\overline{\phi}) + i\mu \, dB_0/d\phi,$$

where
$$\mu = -i\gamma_3(T^* - T_1)/\gamma_4 k_1^2, \quad T^* = (\tfrac{1}{2}k_1^2\gamma_1 - \gamma_2)/\gamma_3. \tag{5.23}$$

We may note here that f_{00}, f_{01}, g_{02} and the adjoint of f_{00} are real while g_{00}, g_{01}, f_{02} and the adjoint of g_{00} are purely imaginary. It follows that γ_1 is purely imaginary and γ_2, γ_3 and γ_4 are real. Consequently T^* is real and, since k_1^2 is pure imaginary, μ is real also. Hence the solution in this region takes the form

$$\overline{\psi} = (\epsilon T)^{\frac{1}{2}} \exp\left\{-\frac{ik_0\overline{\phi}}{\epsilon^{\frac{1}{3}}}\right\}\left[B_0 f_{00} + \epsilon^{\frac{1}{3}}\left(\left(i\mu\frac{dB_0}{\partial\overline{\phi}} - \frac{1}{4}\frac{\gamma_1 k_1^2}{\gamma_4}\right) f_{00} + i\frac{dB_0}{d\overline{\phi}} f_{01}\right) + \epsilon^{\frac{2}{3}}\left(C_1 f_{00} - \left(\mu\frac{d^2B_0}{d\overline{\phi}^2} + \tfrac{1}{4}i\frac{\gamma_1 k_1^2}{\gamma_4}\frac{dB_0}{d\overline{\phi}}\right) f_{01} + \overline{\phi} B_0 f_{02}\right) + \ldots\right] + \text{c.c.} \tag{5.24}$$

We have already shown that the leading terms in this expression match with the mid-latitude solution as $\overline{\phi} \to \infty$, but we need to say something more about the solution for small values of $\overline{\phi}$.

In the flow between concentric cylinders a small increment ϵT_1 above the critical Taylor number T_0 results in a change in the wavenumber proportional to $(\epsilon T_1)^{\frac{1}{2}}$. Here the presence of a secondary flow due to the curvature of the boundaries means that the flow is critical only at $\overline{\phi} = 0$, but one anticipates that in a sufficiently small neighbourhood of $\overline{\phi} = 0$ the effects of the secondary flow are slight and a square-root dependence on T_1 apparent. However, in the narrow region discussed above where $\phi \sim \epsilon^{\frac{2}{3}}$, the dependence on T_1, through the term in μ, is still seen to be linear. This suggests that an even smaller scaling for ϕ is required.

The solution (5.24) contains two scales for ϕ, namely $\epsilon^{\frac{2}{3}}$ and ϵ, and therefore breaks down when $\phi \sim \epsilon$. In that case we write $\phi = \epsilon\lambda$ or equivalently $\overline{\phi} = \epsilon^{\frac{1}{3}}\lambda$ and take λ to be $O(1)$. Then we may seek a solution for $\overline{\psi}$ in the form

$$\overline{\psi} = (\epsilon T)^{\frac{1}{2}} B_{00} \exp\{-ik_0\lambda - i\epsilon^{\frac{1}{2}}\overline{k}_1\lambda\}[f_{00} + \epsilon^{\frac{1}{2}}\overline{k}_1 f_{01} + \epsilon(\lambda^3 f_{20} + \lambda^2 f_{21} + \lambda f_{22} + f_{23}) + \ldots] + \text{c.c.} \tag{5.25}$$

Here $\overline{k}_1$ is a correction to the wavenumber k_0 and is to be determined and B_{00} is a constant. By substituting this expansion into (3.1) and (3.2) and equating terms in $\epsilon\lambda^3$, $\epsilon\lambda^2$, etc., we obtain

$$f_{20} = -\tfrac{1}{6}k_1^2 f_{00}, \quad f_{21} = -\tfrac{1}{2}ik_1^2 f_{01}, \quad f_{22} = f_{02},$$

where k_1^2 is defined by (5.8); the functions f_{00}, f_{01} and f_{02} satisfy (5.1), (5.6) and (5.7) respectively and f_{23} and g_{23} satisfy

$$M_0 g_{23} + iT_0 k_0 f_{23} = 2ik_0 g_{02} - ik_1^2 f_{01} + T_0 f_{02} - ik_0 T_1 f_{00} + 2a_{00} T_0 g_{00}' - 2xk_0^2 g_{00} + 2ik_0 T_0 x f_{00} + \overline{k}_1^2(2k_0 g_{01} - iT_0 f_{01} + g_{00}) + ik_0 T_0 \Omega_{1x} f_{00},$$

$$
\begin{aligned}
M_0^2 f_{23} + 2ik_0(1-x)\,g_{23} = {} & 4ik_0 M_0 f_{20} + 2ik_1^2(M_0 - 2k_0^2) f_{01} + 24ik_0 f_{02} + 2(1-x)\,g_{02} \\
& + 6ik_0 x(1-x)\,g_{00} - 4xk_0^2 M_0 f_{00} + 2ik_0\,\Omega_1 g_{00} \\
& + 2T_0(a_{00}' M_0^2 - a_{00}''') f_{00}' \\
& + \bar{k}_1^2[4k_0 M_0 f_{01} - 2i(1-x)\,g_{01} + 2(M_0 - 2k_0^2) f_{00}].
\end{aligned}
$$

The condition that a solution exists reduces to

$$\bar{k}_1^2 = -(\gamma_3 T_1 + \gamma_2 - k_1^2\gamma_1)/\gamma_4. \tag{5.26}$$

The square-root dependence of $\bar{k}_1$ on T_1 is now clear. In order that the solution in the neighbourhood of the equator is oscillatory we require $\bar{k}_1$ to be real and therefore, since γ_3/γ_4 turns out to be negative,

$$T_1 \geqslant T_{1c} = (k_1^2\gamma_1 - \gamma_2)/\gamma_3. \tag{5.27}$$

Numerical computation of all the quantities involved in T_{1c} gives

$$\gamma_1 = -1{\cdot}95\times10^{-6}i, \quad \gamma_2 = -5{\cdot}94\times10^{-3}, \quad \gamma_3 = -3{\cdot}21\times10^{-6}, \quad T_{1c} = -1860.$$

It remains to verify that this solution matches with that given in (5.24) as $\bar{\phi} \to 0$. If $\bar{\phi} \ll 1$ one solution of (5.14) is

$$B_0(\bar{\phi}) = B_{00}(1 - \tfrac{1}{6}k_1^2\bar{\phi}^3 + \ldots)$$

and the corresponding solution of (5.22) is

$$C_0(\bar{\phi}) = B_{00}(\gamma_3 T_1 + \gamma_2 - k_1^2\gamma_1)(\bar{\phi}^2 + \ldots)/2\gamma_4.$$

When $\bar{\phi} = \epsilon^{\frac{1}{3}}\lambda$ this means that $\bar{\psi}$ becomes

$$
\begin{aligned}
\bar{\psi} &= (\epsilon T)^{\frac{1}{2}} B_{00}\exp\{-ik_0\lambda\}\,[f_{00}(1 - \tfrac{1}{2}\epsilon\bar{k}_1^2\lambda - \tfrac{1}{6}\epsilon k_1^2\lambda^3 + \ldots) \\
&\quad - f_{01}(\tfrac{1}{2}i\epsilon k_1^2\lambda^2 + i\epsilon\bar{k}_1^2\lambda + \ldots) + \epsilon\lambda f_{02} + \ldots] + \text{c.c.} \\
&= (\epsilon T)^{\frac{1}{2}} B_{00}\exp\{-ik_0\lambda\}\{\cos(\epsilon^{\frac{1}{2}}\bar{k}_1\lambda)\,[(1 - \tfrac{1}{6}\epsilon k_1^2\lambda^3 + \ldots)f_{00} - \tfrac{1}{2}ik_1^2\lambda^2 f_{01} + \lambda f_{02} + \ldots] \\
&\quad + i\epsilon^{\frac{1}{2}}\bar{k}_1\sin(\epsilon^{\frac{1}{2}}\bar{k}_1\lambda)f_{01} + \ldots\} + \text{c.c.}
\end{aligned}
$$

This expression is equivalent to that obtained by adding together the solutions corresponding to the two roots for $\bar{k}_1$ given by (5.26). A similar argument holds for the second solution of (5.14) for $B_0(\bar{\phi})$ and hence for the combination of solutions used in (5.15).

Our discussion of the flow in the neighbourhood of the equator is now complete. In summary, we have found that in a region of thickness $\epsilon^{\frac{1}{3}}$ the singularity in $\bar{\psi}$ and $\bar{\Omega}$ at $\phi = 0$ is removed and the solution is given in terms of Airy functions. The secondary flow is still a dominating influence and it is only in an inner region of thickness ϵ that the effect of T_1 becomes as important.

T_1 is still undetermined except that it should be greater than T_{1c}. In the next section we satisfy the boundary conditions at the poles $\theta = 0, \pi$ and this allows T_1 to take any of a discrete set of values. A complication arises because another new scaling is required for θ near the poles and this is considered first.

6. The solution near the poles $\theta = 0, \pi$

The expansion (3.5) in ϵ breaks down when $\tan\theta$ is $O(\epsilon)$ because of the term in $\cot\theta\,\partial/\partial\theta$ in $\tilde{D}^2$ [see (2.3)]. Near the poles $\theta = 0, \pi$ a new scaling and a new solution are necessary and the latter is then matched with the solution valid in mid-latitudes presented in § 4 as $\theta \to 0, \pi$.

At $\theta = 0$, (4.1) reduces to

$$Mg_0 = 0, \quad M^2 f_0 - (1-x)g_0' + g_0 = 0, \tag{6.1}$$

with

$$f_0 = f_0' = g_0 = 0 \quad \text{at} \quad x = 0, 1. \tag{6.2}$$

There is a spectrum of eigenvalues of this system of the form $k = in\pi$, where n is an integer, of which the first pair ($n = 1$) matches with $(k^{(5)}, k^{(6)})$. There are, however, other complex eigenvalues given by

$$\sinh k = \pm k, \tag{6.3}$$

whose eigensolutions are

$$f_0 = (1-x)\sinh kx \mp x \sinh k(1-x), \quad g_0 \equiv 0. \tag{6.4}$$

The computed values of $k^{(1)}$ and $k^{(2)}$ at $\theta = 0$ are $2{\cdot}25074 \mp i4{\cdot}21239$; they are related to the first eigenvalue of (6.3) (Hillman & Saltzer 1943).

Hence, as $\theta \to 0$,

$$\left.\begin{aligned} \overline{\psi} &\sim (\epsilon T)^{\frac{1}{2}} \sum_{j=1}^{6} \bar{A}_0^{(j)}(0)\,(\sin\theta)^{\frac{1}{2}} f_0^{(j)}(0,x) \exp\{I_j(0)\} \exp\{ik^{(j)}(0)\,\theta/\epsilon\}, \\ \overline{\Omega} &\sim \sum_{j=5}^{6} \bar{A}_0^{(j)}(0)\,(\sin\theta)^{\frac{1}{2}} \sinh(k^{(j)}(0)\,x) \exp\{I_j(0)\} \exp\{ik^{(j)}(0)\,\theta/\epsilon\}, \end{aligned}\right\} \tag{6.5}$$

where

$$\left.\begin{aligned} &I_j(\theta) = \frac{i}{\epsilon}\int_{\frac{1}{2}\pi}^{\theta} k^{(j)}(\theta)\,d\theta \quad (j = 1, \ldots, 6), \\ &k^{(4)}(0) = -k^{(1)}(0), \quad k^{(3)}(0) = -k^{(2)}(0) = -\bar{k}^{(1)}(0), \quad k^{(6)}(0) = -k^{(5)}(0) = i\pi. \end{aligned}\right\} \tag{6.6}$$

Near $\theta = 0$ we rescale by writing $\theta = \epsilon y$ with $y = O(1)$. Then

$$\tilde{D}^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{1}{y}\frac{\partial}{\partial y} = \tilde{D}_0^2, \quad \text{say},$$

to a first approximation in ϵ. $\overline{\psi}$ and $\overline{\Omega}$ now satisfy

$$\tilde{D}_0^2\overline{\Omega} = \tilde{D}_0^4\overline{\psi} - 2[(1-x)\,\partial\overline{\Omega}/\partial x - \overline{\Omega}] = 0. \tag{6.7}$$

The solution of (6.7) which matches with (6.5) as $y \to \infty$ is

$$\left.\begin{aligned} \overline{\psi} &= (\epsilon T)^{\frac{1}{2}} \sum_{j=1}^{6} c_j f_0^{(j)}(0,x)\, y H_1^{(1)}(k^{(j)}(0)\,y), \\ \overline{\Omega} &= \sum_{j=5}^{6} c_j \sinh(k^{(j)}(0)\,x)\, y H_1^{(1)}(k^{(j)}(0)\,y), \end{aligned}\right\} \tag{6.8}$$

where $H_1^{(1)}$ is the Hankel function of the first kind and order one and

$$c_j = \epsilon^{\frac{1}{2}}(\tfrac{1}{2}\pi k^{(j)}(0))^{\frac{1}{2}} e^{\frac{3}{4}\pi i} \exp\{I_j(0)\}\,\bar{A}_{0j}(0) \quad (j = 1, \ldots, 6). \tag{6.9}$$

Using the relations between the eigenvalues (6.6) and the corresponding eigenfunctions, $\overline{\psi}$ may be written as

$$\begin{aligned}\overline{\psi} &= (\epsilon T)^{\frac{1}{2}} y\{f_0^{(1)}(0,x)\,[c_1 H_1^{(1)}(k^{(1)}(0)\,y)+c_4 H_1^{(1)}(-k^{(1)}(0)\,y)] \\ &\quad + f_0^{(2)}(0,x)\,[c_2 H_1^{(1)}(k^{(2)}(0)\,y)+c_3 H_1^{(1)}(-k^{(2)}(0)\,y)] \\ &\quad + i\sin\pi x[c_5 H_1^{(1)}(i\pi y)+c_6 H_1^{(1)}(-i\pi y)]\} \\ &= (\epsilon T)^{\frac{1}{2}} y\{f_0^{(1)}(0,x)\,[(c_1+c_4)\,J_1(k^{(1)}(0)\,y)+i(c_1-c_4)\,Y_1(k^{(1)}(0)\,y)] \\ &\quad + f_0^{(2)}(0,x)\,[(c_2+c_3)\,J_1(k^{(2)}(0)\,y)+i(c_2-c_3)\,Y_1(k^{(2)}(0)\,y)] \\ &\quad + i\sin\pi x[(c_5+c_6)\,J_1(i\pi y)+i(c_5-c_6)\,Y_1(i\pi y)]\}. \end{aligned} \tag{6.10}$$

In order that the velocity components be finite at $y = 0$ we require $\overline{\psi} \sim y^2$ as $y \to 0$, which means that

$$c_1 = c_4, \quad c_2 = c_3, \quad c_5 = c_6.$$

Together with (6.9) and (4.9), the last condition contradicts (5.10) and means that

$$c_5 = c_6 = \alpha_5 = \alpha_6 = 0,$$

which confirms our earlier assertion that modes 5 and 6 play no part to this order in ϵ. From (6.9) the first two conditions give

$$\exp\{I_1(0)\}\,\bar{A}_{01}(0) = e^{\frac{1}{2}\pi i}\exp\{I_4(0)\}\,\bar{A}_{04}(0) = e^{\frac{1}{2}\pi i}\exp\{\tilde{I}_2(0)\}\,\tilde{\bar{A}}_{04}(0). \tag{6.11}$$

Hence we require $\quad \operatorname{Im}\{e^{\frac{3}{2}\pi i}\exp\{I_1(0)+I_2(0)\}\,\bar{A}_{01}(0)\,\bar{A}_{02}(0)\} = 0.$

Using (4.9) and (5.18) we obtain

$$\operatorname{Im}\{\alpha_1'\alpha_2' e^{\frac{3}{2}\pi i}\exp\{I_1(0)+I_2(0)\} - \exp\int_0^{\frac{1}{2}\pi}(\chi^{*(1)}+\chi^{*(2)})\,d\theta\} = 0$$

and since α_1' and α_2' are real numbers we have

$$\frac{1}{\epsilon}\int_0^{\frac{1}{2}\pi}[\operatorname{Re}(k^{(1)}+k^{(2)})+\epsilon\operatorname{Im}(\chi^{*(1)}+\chi^{*(2)})]\,d\theta = (m-\tfrac{3}{4})\pi, \tag{6.12}$$

where m is an integer.

7. Results and conclusions

The main object of this paper is to determine the critical value of the Taylor number T at which instability first occurs. When $\epsilon \ll 1$ we write

$$T = T_0 + \epsilon T_1 + \dots.$$

Since the azimuthal velocity of the inner sphere is greatest at the equator we expect that it is there that the flow is most unstable. In the immediate neighbourhood of the equator, when $\phi = O(\epsilon)$, we have found that to a first approximation the flow is equivalent to that between concentric cylinders. On this scale the north and south poles are at infinity and we require only that the wavenumber be real. The minimum value of T for which the wavenumber is real is known to be 1694·95 and we take this as our value for T_0. Furthermore, T_1 produces a correction to the wavenumber proportional to $\epsilon^{\frac{1}{2}}$ and the condition that this is real also is $T_1 \geqslant T_{1c} = -1860$.

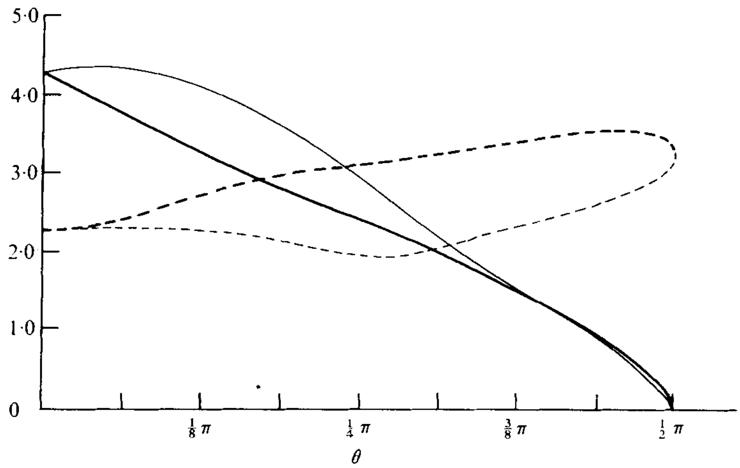

FIGURE 1. The computed values of $k^{(1)}$ and $k^{(2)}$ as functions of θ for $\sigma = 0$ and $T_0 = 1694{\cdot}95$. – – –, Re $k^{(1)}$; ——, $-\text{Im}\, k^{(1)}$; ---, Re $k^{(2)}$; ——, Im $k^{(2)}$.

Then the solution may be matched with that in an outer region centred on $\phi = 0$ when ϕ is $O(\epsilon^{\frac{2}{3}})$ and through that region to the mid-latitude region, where ϕ is $O(1)$. The solution is continued to the poles through narrow regions where θ or $\pi - \theta$ is $O(\epsilon)$.

The fluid velocity is finite at the poles provided that

$$\int_0^{\frac{1}{2}\pi} [\text{Re}\,(k^{(1)} + k^{(2)}) + \epsilon\, \text{Im}\,(\chi^{*(1)} + \chi^{*(2)})]\, d\theta = \epsilon(m - \tfrac{3}{4})\,\pi. \tag{7.1}$$

Here $k^{(1)}$ and $k^{(2)}$ are the eigenvalues of (4.1) and (4.2) that have positive real part; they have been computed numerically for several values of θ for $T_0 = 1694{\cdot}95$ and are shown in figure 1. To a first approximation in ϵ, (7.1) reduces to

$$\int_0^{\frac{1}{2}\pi} \text{Re}\,\{k^{(1)} + k^{(2)}\}\, d\theta = \epsilon m\pi.$$

Since the left-hand side is fully determined when T_0 is given and ϵ is a vanishingly small parameter this equation merely says that m is a large $[O(\epsilon^{-1})]$ number and the equation is satisfied arbitrarily closely even when m is an integer. $\chi^{*(1)}$ and $\chi^{*(2)}$ have also been computed; the results give

$$\int_0^{\frac{1}{2}\pi} \text{Re}\,\{k^{(1)} + k^{(2)}\}\, d\theta = 8{\cdot}267,$$

$$\int_0^{\frac{1}{2}\pi} \text{Im}\,(\chi^{*(1)} + \chi^{*(2)})\, d\theta = 6{\cdot}78 + 8{\cdot}58 \times 10^{-4} T_1,$$

and (7.1) is therefore satisfied if

$$T_1 = 10^4[(m - \tfrac{3}{4})\,\pi - 8{\cdot}267/\epsilon - 6{\cdot}78]/8{\cdot}58. \tag{7.2}$$

For each integer value of m there is a corresponding value of T_1; the spectrum of possible values of T_1 is therefore discrete. The critical value of T_1 is taken to be the least such value of T_1 subject to the constraint $T_1 \geqslant T_{1c} = -1860$.

One could regard the correction to T_0 as being due to two separate causes. First, the correction due to the flow in the immediate neighbourhood of the equator decreases

T by an amount 1860ϵ. Second, T must be given an increment to ensure that the global condition (7.2) is satisfied. In geometries where the flow becomes simultaneously unstable at all points (for example, Bénard convection in a box; see Hall & Walton 1977) the latter correction is only $O(\epsilon^2)$ because the increment in wavenumber is proportional to the square root of the increment in the Taylor number. Here the increments in the wavenumbers $\chi^{(1)*}$ and $\chi^{(2)*}$ vary linearly with T_1 over the entire range of θ except very close to $\theta = \frac{1}{2}\pi$. Consequently this correction is $O(\epsilon)$ also and therefore is more important than might perhaps have been anticipated.

Of course, for certain values of ϵ, $T_1 = -1860$ will satisfy (7.2) exactly for some integer value of m, but such values of ϵ are exceptional. For example, when $\epsilon = 0{\cdot}1$ we find that the critical value of T_1 is -820 with $m = 29$ and the critical value of T is 1613. When $\epsilon = 10^{-2}$ the corresponding figures are -210, 266 and 1692·9.

For $\epsilon = 0{\cdot}0527$ we find the critical value of T_1 to be 590 with $m = 53$ and the critical value of T to be 1726. This agrees fairly well with Wimmer's (1976) experimental result of 1705·7 with experimental errors as large as $\pm 113{\cdot}7$. Wimmer also reports that the critical value of T can be either greater or less than T_0 and this agrees with our findings.

The integer m is a rough guide to the number of cells that are fitted between the two poles, but because their amplitudes decay rapidly with latitude most will not be visible. In the $\epsilon^{\frac{2}{3}}$ region near the equator the e-folding distance for ϕ is $O(\epsilon^{\frac{2}{3}})$ and on this basis we might expect about $\epsilon^{-\frac{1}{3}}$ cells to be visible. For only moderately small values of ϵ this suggests that very few cells will actually appear and this is borne out by the experiments of Wimmer (1976).

Finally we may examine the amplitudes of the two modes $j = 1, 2$ which make up the solution. The magnitudes of the eigensolutions $f_0^{(1)}(\theta, x)$ and $f_0^{(2)}(\theta, x)$ are typically $O(1)$, so that the ratio ρ of the amplitudes of the two modes is

$$\rho = \frac{\bar{A}_{02} \exp \int_{\theta}^{\frac{1}{2}\pi} \epsilon^{-1} \operatorname{Im}\{k^{(2)}\}\, d\theta}{\bar{A}_{01} \exp \int_{\theta}^{\frac{1}{2}\pi} \epsilon^{-1} \operatorname{Im}\{k^{(1)}\}\, d\theta}. \tag{7.3}$$

From (6.11) we have

$$\frac{\bar{A}_{02}}{\bar{A}_{01}} = \frac{\exp \int_{0}^{\frac{1}{2}\pi} \epsilon^{-1} \operatorname{Im}\{k^{(1)}\}\, d\theta}{\exp \int_{0}^{\frac{1}{2}\pi} \epsilon^{-1} \operatorname{Im}\{k^{(2)}\}\, d\theta} \tag{7.4}$$

and combining (7.3) and (7.4) we have

$$\rho = \exp \int_0^{\theta} \epsilon^{-1} \operatorname{Im}\{k^{(1)} - k^{(2)}\}\, d\theta.$$

Hence the amplitudes of the two modes are of similar orders of magnitude when $\theta = 0$, i.e. at the pole, but at the equator $\theta = \frac{1}{2}\pi$ that mode which grows with latitude (index 2) is smaller than the mode which decays (index 1) by an exponential factor.

The author wishes to thank Professor J. T. Stuart for some helpful discussions.

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